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SHORT COMMUNICATION

ON NORMAL FLOW BOUNDARY CONDITIONS IN FINITE ELEMENT CODES FOR TWO-DIMENSIONAL SHALLOW WATER FLOW

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INTRODUCTION

In a recent paper, Engelman *et al.*[1] have reviewed various techniques for implementing specified normal flow boundary conditions in finite element codes. One of the biggest problems in specifying this type of boundary condition is that, at a node, the normal direction along the boundary of the finite element grid may not be unique. Engelman *et al.* confirm the two-dimensional derivation of Gray[2] which selects an average normal at such nodes using a mass conservation criteria and extend that work to three dimensions.

In two-dimensional shallow water problems, a typical boundary condition is the specification of the normal velocity or flux. For example at a land boundary, the normal flow is zero but the flow is allowed to slip tangentially along the boundary. If node j is such a boundary node, this boundary condition is implemented in the momentum equations by rotating the $x-y$ equations weighted with respect to the basis function at node j into tangential and normal equations, dropping the normal equation in favour of the boundary condition, and solving the tangential equation.[3] In such simulations, difficulties discussed by Engelman *et al.* may arise in selection of the normal direction. These difficulties arise, in part, because the vertically integrated mass conservation equation is more complex than the conservation equation for an incompressible fluid. It is our purpose here to examine instances when additional considerations are required for selection and application of the normal direction conditions.

COMPUTATION OF THE NORMAL DIRECTION

For tidal simulations performed using quadratic basis functions, King[4] has proposed a method which ensures that the boundary of a finite element grid will be smooth and thus have a unique normal direction at every point. However, if linear elements are used or if one does not wish to perform the calculations required to guarantee smoothness, the normal direction at a corner node will not necessarily be unique. In these instances it is necessary to calculate an appropriate normal direction for use in computations. The appropriate direction does not depend only on the grid being used but also on the way the dependent variables are approximated.

Tidal or shallow-water simulations are concerned with obtaining solutions for the total depth, H, and for the vertically averaged x- and y-components of velocity, denoted U and

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V, respectively. Sometimes these problems are formulated in terms of the surface elevation, ζ, which is equal to $H-h$ where h is the bathymetry or distance from mean sea level to the bottom of the water body. Also the water fluxes, $Q_x = UH$ and $Q_y = VH$, are sometimes solved for instead of the velocities. This last choice of dependent variables does have some bearing on the normal direction computation.

Case I: normal direction for Q_x and Q_y as dependent variables

If the water flux $\mathbf{Q}$ is expanded in terms of basis functions such that

$$\mathbf{Q} = \sum_{i=1}^{N} \mathbf{Q}_i \phi_i \tag{1}$$

where $\phi_i(x, y)$ are the basis functions and $\mathbf{Q}_i$ is the estimate of $\mathbf{Q}$ at node i, then for the region Ω under consideration, the net outward flux F is defined by

$$F = \int_{\partial\Omega} \mathbf{Q}\cdot\mathbf{n}\,\mathrm{d}S = \sum_{i=1}^{N}\left[Q_{x_i}\int_{\partial\Omega}\phi_i n_x\,\mathrm{d}S + Q_{y_i}\int_{\partial\Omega}\phi_i n_y\,\mathrm{d}S\right] \tag{2}$$

Of course because ϕ_i is zero on $\partial\Omega$ unless node i is a boundary node, the flux can be computed by summing only over the N_B boundary nodes. We may now apply the divergence theorem to (2) and obtain

$$F = \int_{\Omega} \nabla\cdot\mathbf{Q}\,\mathrm{d}\Omega = \sum_{i=1}^{N_B}\left[Q_{xi}\int_{\Omega}\frac{\partial\phi_i}{\partial x}\mathrm{d}\Omega + Q_{y_i}\int_{\Omega}\frac{\partial\phi_i}{\partial y}\mathrm{d}\Omega\right] \tag{3}$$

If the desired but unknown nodal components of the effective normal at node i are defined such that

$$Q_{x_i} = n_{x_i}Q_{n_i} - n_{y_i}Q_{t_i} \tag{4a}$$

$$Q_{y_i} = n_{y_i}Q_{n_i} + n_{x_i}Q_{t_i} \tag{4b}$$

where n_{x_i} and n_{y_i} are the x- and y-components of $\mathbf{n}$ at node i and Q_{n_i} and Q_{t_i} are the normal and tangential components of $\mathbf{Q}$ at node i, then substitution of (4) into (3) yields

$$F = \sum_{i=1}^{N_B}\left[Q_{n_i}\left(n_{x_i}\int_{\Omega}\frac{\partial\phi_i}{\partial x}\mathrm{d}\Omega + n_{y_i}\int_{\Omega}\frac{\partial\phi_i}{\partial y}\mathrm{d}\Omega\right) + Q_{t_i}\left(n_{x_i}\int_{\Omega}\frac{\partial\phi_i}{\partial y}\mathrm{d}\Omega - n_{y_i}\int_{\Omega}\frac{\partial\phi_i}{\partial x}\mathrm{d}\Omega\right)\right] \tag{5}$$

The net outward flux F should be independent of the tangential fluxes for all possible combinations of Q_{t_i}. Therefore the coefficient of Q_{t_i} in (5) must be zero for all i or

$$n_{x_i}\int_{\Omega}\frac{\partial\phi_i}{\partial y}\mathrm{d}\Omega = n_{y_i}\int_{\Omega}\frac{\partial\phi_i}{\partial x}\mathrm{d}\Omega \tag{6}$$

Then, because $n_{x_i}^2 + n_{y_i}^2 = 1$, we obtain

$$n_{x_i} = \frac{1}{S_i}\int_{\Omega}\frac{\partial\phi_i}{\partial x}\mathrm{d}\Omega \tag{7a}$$

$$n_{y_i} = \frac{1}{S_i}\int_{\Omega}\frac{\partial\phi_i}{\partial y}\mathrm{d}\Omega \tag{7b}$$

where

$$S_i = \left[\left(\int_{\Omega}\frac{\partial\phi_i}{\partial x}\mathrm{d}\Omega\right)^2 + \left(\int_{\Omega}\frac{\partial\phi_i}{\partial y}\mathrm{d}\Omega\right)^2\right]^{1/2} \tag{8}$$

These results are the same as those of Gray[2] and Engelman *et al.*[1] and the derivation is virtually identical to that of Engelman *et al.*

Case II: normal direction for U and V as independent variables

If instead of expanding the water flux $\mathbf{Q}$, we expanded the water velocity $\mathbf{v}$ in terms of the basis functions such that

$$\mathbf{v} = \sum_{i=1}^{N} \mathbf{v}_i \phi_i \tag{9}$$

then the net outward flux F is defined by noting that $\mathbf{Q} = H\mathbf{v}$ such that (2) becomes

$$F = \int_{\partial\Omega} H\mathbf{v} \cdot \mathbf{n}^* \, \mathrm{d}S = \sum_{i=1}^{N_B} \left[U_i \int_{\partial\Omega} H\phi_i n_x^* \, \mathrm{d}S + V_i \int_{\partial\Omega} H\phi_i n_y^* \, \mathrm{d}S \right] \tag{10}$$

Here we have replaced N by N_B, the set of boundary node numbers as discussed previously. For the present we will say nothing of how H is represented in space. Application of the divergence theorem to (10) yields

$$F = \int_{\Omega} \nabla \cdot (H\mathbf{v}) \, \mathrm{d}\Omega = \sum_{i=1}^{N_B} \left[U_i \int_{\Omega} \frac{\partial (H\phi_i)}{\partial x} \mathrm{d}\Omega + V_i \int_{\Omega} \frac{\partial (H\phi_i)}{\partial y} \mathrm{d}\Omega \right] \tag{11}$$

If equations (4) are now invoked and we let $U_i = Q_{x_i}/H_i$ and $V_i = Q_{y_i}/H_i$ then equation (11) becomes

$$F = \sum_{i=1}^{N_B} \left[Q_{n_i} \left(\frac{n_{x_i}^*}{H_i} \int_{\Omega} \frac{\partial (H\phi_i)}{\partial x} \mathrm{d}\Omega + \frac{n_{y_i}^*}{H_i} \int_{\Omega} \frac{\partial (H\phi_i)}{\partial y} \mathrm{d}\Omega \right) + Q_{t_i} \left(\frac{n_{x_i}^*}{H_i} \int_{\Omega} \frac{\partial (H\phi_i)}{\partial y} \mathrm{d}\Omega - \frac{n_{y_i}^*}{H_i} \int_{\Omega} \frac{\partial (H\phi_i)}{\partial x} \mathrm{d}\Omega \right) \right] \tag{12}$$

Again the net outward flux F should be independent of the tangential fluxes, so the coefficient of each Q_{t_i} must be zero or

$$n_{x_i}^* \int_{\Omega} \frac{\partial (H\phi_i)}{\partial y} \mathrm{d}\Omega = n_{y_i}^* \int_{\Omega} \frac{\partial (H\phi_i)}{\partial x} \mathrm{d}\Omega \tag{13}$$

By requiring the magnitude of the normal to be unity we obtain

$$n_{x_i}^* = \frac{1}{S_i^*} \int_{\Omega} \frac{\partial (H\phi_i)}{\partial x} \mathrm{d}\Omega \tag{14a}$$

$$n_{y_i}^* = \frac{1}{S_i^*} \int_{\Omega} \frac{\partial (H\phi_i)}{\partial y} \mathrm{d}\Omega \tag{14b}$$

where

$$S_i^* = \left[\left(\int_{\Omega} \frac{\partial (H\phi_i)}{\partial x} \mathrm{d}\Omega \right)^2 + \left(\int_{\Omega} \frac{\partial (H\phi_i)}{\partial y} \mathrm{d}\Omega \right)^2 \right]^{1/2} \tag{15}$$

These results agree with those in (7) and (8) only for the special case of H independent of space. In most practical problems, H is a function of time as well as space and therefore the normal components would vary with time and would have to be recomputed at each time step. If most of the spatial variation of H is due to bathymetry rather than surface elevation, a time invariant estimation for $n_{x_i}^*$ and $n_{y_i}^*$ may be obtained by substituting h for H in (14) and (15). For purposes of computational efficiency, it may be useful to note that if H is known through an expansion in terms of ϕ_i, one only need consider the terms in the expansion which involve the boundary nodes. Thus if

$$H = \sum_{j=1}^{N} H_j \phi_j \tag{16}$$

then (14) and (15) become

$$n^*_{x_i} = \frac{1}{S^*_i} \sum_{j=1}^{N_B} \left[H_j \int_\Omega \frac{\partial(\phi_i \phi_j)}{\partial x} \, d\Omega \right] \tag{17a}$$

$$n^*_{y_i} = \frac{1}{S^*_i} \sum_{j=1}^{N_B} \left[H_j \int_\Omega \frac{\partial(\phi_i \phi_j)}{\partial y} \, d\Omega \right] \tag{17b}$$

and

$$S^*_i = \left\{ \left(\sum_{j=1}^{N_B} \left[H_j \int_\Omega \frac{\partial(\phi_i \phi_j)}{\partial x} \, d\Omega \right] \right)^2 + \left(\sum_{j=1}^{N_B} \left[H_j \int_\Omega \frac{\partial(\phi_i \phi_j)}{\partial y} \, d\Omega \right] \right)^2 \right\}^{1/2} \tag{18}$$

Although the computation of $n^*_{x_i}$ and $n^*_{y_i}$ requires more work in case II than the computation of n_{x_i} and n_{y_i} in case I, both derivations are straightforward and the actual calculations can be easily made in a computer code. The components of $\mathbf{n}$ (or $\mathbf{n}^*$) obtained may be used to rotate the momentum equations into normal and tangential co-ordinates and apply the normal velocity boundary condition. After solution of the tangential momentum equations, the velocities may be rotated back into x-y co-ordinates, again using the components of the normal. The solution of the momentum equation is thus accomplished by knowing the appropriate unit normal vector, regardless of whether case I or case II is considered.

EXAMPLE CALCULATION

Assume that we are performing calculations on a linear finite element grid. On the boundary of the grid are three nodes labelled A, B, and C in counterclockwise order where the depths are H_A, H_B and H_C, respectively (Figure 1). The length of the element side joining B and C is a, and of the side joining A and B is c. Nodes A and C are separated by a distance b. We can use equations (17) and (18) to get the case II values of $n^*_{x_B}$ and $n^*_{y_B}$ and note that when $H_A = H_B = H_C$, the solutions will become identical to those for case I. The algebra involved is somewhat lengthy but it is easy to show that

$$n^*_{x_B} = \frac{1}{S^*} \left[\alpha(y_C - y_A) + \left(\alpha - \frac{1}{\alpha} \right)(y_A - y_B) \right] \tag{19a}$$

$$n^*_{y_B} = \frac{1}{S^*} \left[\frac{1}{\alpha}(x_A - x_C) + \left(\alpha - \frac{1}{\alpha} \right)(x_B - x_C) \right] \tag{19b}$$

$$S^* = b \left[1 + \left(\frac{a}{b} \right)^2 (\alpha^2 - 1) + \left(\frac{c}{b} \right)^2 \left(\frac{1}{\alpha^2} - 1 \right) \right]^{1/2} \tag{20}$$

where

$$\alpha = \left[\frac{2H_B + H_C}{2H_B + H_A} \right]^{1/2} \tag{21}$$

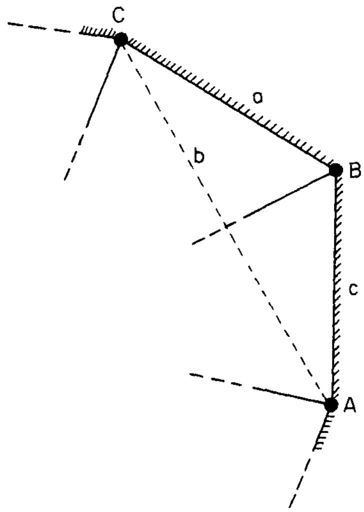

Figure 1. Boundary section of a linear finite element grid

The components of **n** for the case I situation can be obtained from (19) and (20) by setting α equal to unity. To obtain the change in normal direction between the case I and case II formulation we dot **n** with **n*** such that

$$\mathbf{n} \cdot \mathbf{n}^* = \cos\theta = \frac{1+\alpha^2+\left(\frac{c}{b}\right)^2(1-\alpha^2)+\left(\frac{a}{b}\right)^2(\alpha^2-1)}{2\left[\alpha^2+\left(\frac{c}{b}\right)^2(1-\alpha^2)+\left(\frac{a}{b}\right)^2(\alpha^4-\alpha^2)\right]^{1/2}} \tag{22}$$

where θ is the angle between **n** and **n***. When $H_C = H_A$, α will equal 1, $\cos\theta$ will also be unity and **n** will be equal to **n***. When H_C is not equal to H_A, α will differ from unity, and equation (22), along with the finite element geometry, may be used to investigate the change in normal direction induced by flow depth effects in the case II analysis.

CONCLUSION

This paper has shown that in surface water modelling the method for selecting the appropriate normal direction at a finite element boundary node depends upon the grid, and also on whether the basis function expansion is performed on velocity or flux. It has been shown that when the expansion is in terms of velocity, the appropriate normal direction at a node may vary in time. However for most applications, this variation is expected to be small.

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